Institute of Actuaries

Students' Society

Valuation of Liabilities and

Distribution of Surplus

(Notes and papers...)

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AND

Distribution of Surplus

NOTES, PAPERS AND DISCUSSION
ARRANGED BY THE INSTITUTE OF
ACTUARIES STUDENTS' SOCIETY

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56, FARRINGDON STREET, E.C.

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Valuation of Liabilities and Distribution of Surplus.

Valuations of liabilities are made at the present time to ascertain the actual amount of reserve which it is necessary for an insurance company to hold to enable it to fulfil its obligations and maintain its rate of future bonuses, and so decide what portion of the assets can properly be distributed as bonus. In obtaining a valuation of liabilities a student should bear in mind that there is a very large difference between the reserve required for an insurance company to be solvent and the reserve which it is advisable to keep in hand so as to enable the office to earn a reasonably level amount of profit. A reserve for solvency would be found by subtracting (*a*) the value of the future premiums receivable by the office, less the value of future expenses, calculated at the prevailing rate of interest* and by tables representing the rate of mortality to be expected in the future, from (*b*) the value of the sums assured and existing bonus calculated in a similar way.

The student will soon satisfy himself that there is a large margin between solvency and the assets of the insurance companies, but he will see that a part of this is necessary to give stability and to equalise bonuses. The amount required

* Rates of interest mean rates clear of tax: the rate actually used would probably be the next quarter per cent. less than the present rate: it is an approximation to the rate that can be reasonably anticipated.

for the latter purpose can be ascertained by making a valuation, on the bases mentioned in the preceding paragraph, assuming that the rate of bonus it is proposed to declare should be continued in future. This method has sometimes been called a bonus reserve valuation.

With regard to the tables of mortality to be used, modern practice would generally assume the O^M tables although for various reasons some offices have adopted the $O^{M(5)}$ tables and more recently the use of select tables has been discussed. An aggregate table of mortality gives a reasonable approximation to a select table valuation, in the case of an office doing a steady new business, so far as sum assured is concerned and it also gives a similar approximation for the value of office premiums (*see* D. C. Fraser, J.I.A., xl., pp. 122, *et seq.*), but the valuation by an aggregate table of a net premium calculated from an aggregate table will not enable us to obtain a select valuation. The use however of the net premium method of valuation means that the loading reserve to cover future expenses and contributions to profits is varied automatically according to the table used in the valuation, and in any office adopting the net premium method the policy reserves may differ widely from those of a select table valuation, the result being that loading profit and mortality profit are to some extent interchanged. It has been shown by experience and has also been indicated in various papers that a net premium valuation will give a reasonable allowance for equalising bonuses if the bonus is distributed on the simple or compound reversionary plan and that the amount of bonus that can be distributed according to them depends to some extent on the difference between the rate of interest used in the valuation and that actually earned by the office. Thus an office making an O^M 3 per cent. net premium valuation and earning 4 per cent.

could give a £2 simple or 27*s*. compound reversionary bonus and would have sufficient in hand to enable it to keep up these bonuses.*

Of the papers recommended for a first reading the one by Sir Gerald Ryan (J.I.A., xxxviii., 69) is not reprinted, as the number of the Journal in which it appears is still in print. The other papers on the distribution of surplus are printed in the following pages. In the paper by Mr. Lidstone an analysis is made of the results obtained by distributing surplus according to Dr. Sprague's contribution plan which endeavours to allow for the source from which the profit has been obtained and Mr. Browne's letter gave the actual results of a somewhat similar method of distribution which had been employed by him with success. Mr. Lidstone's paper shows that the compound reversionary bonus plan gives a reasonable approximation to the results that would be obtained by the more elaborate method devised by Dr. Sprague when average premiums are charged, but it does not of course follow that in extreme cases the two methods will give the same results.

A point which is outside the scope of these papers is the consideration of cases in which the contribution method cannot give a correct distribution of bonus. An example is the well-known case in which the premiums charged assume that the bonus is given not according to the number of years in each valuation period but according to the number of years that have elapsed at the date of valuation since the policy was effected. In this case the premiums are relatively high at the younger ages of entry and relatively low at the older ages, and as the bonus increases very rapidly a heavy reserve has to be kept in order to provide for the large

* This general rule applies to a normal office in normal times. Exceptional losses, *e.g.*, depreciation, extra mortality, &c., might upset such general rules.

bonuses payable in the future. Other examples are to be found where bonuses are given by reduction of premium or are deferred. The necessary reserve would be essentially different from the reserve that could be obtained by an ordinary net premium valuation, and some form of Bonus Reserve Valuation is necessary to give a clear view of the position.

In such a case no contribution method that has yet been devised can enable the actuary to decide what amount it is necessary for him to carry forward before dividing the bonus and to this extent contribution methods fail.

The student will notice in reading Mr. Lidstone's paper that he discusses the value of future allotments of surplus and in this connection gives (§ 36) an interesting result of which an algebraic proof will be found in Appendix B to his paper. Attention may also be drawn to the same writer's paper on changes in pure premium policy values (J.I.A., xxxix., 209) in which general results are reached that can be interpreted in connection with division of surplus, and to a letter published in J.I.A., xlvi., 202, in which Mr. Lidstone drew attention to the application of these results to the case of sickness benefits or a combination of sickness and other benefits and gave the following note : " Both these " investigations extended to the case where the sum assured " varied from year to year, and both provided for p and q " being entirely independent, these quantities being defined " as follows :—

p_n = probability that if the status be in existence at the beginning of the nth year, it will survive to the end of the year.

q_n = probability that if the status be in existence at the beginning of the nth year, the sum assured, S_n, will become payable at the end of the year.

" Thus $(q_n S_n)$ is the expected amount of claim during the " year, and it will be found that it always enters into the " investigations as one quantity, which might have been " represented by a single symbol, say β_n defined as the " expected claim at the end of the nth year for each status in " force at the beginning of the year. If, therefore, β " represents the expected cost of sickness, or the expected " cost of sickness plus death benefit plus any other benefit, " the whole of the above general investigations will still " apply, practically without alteration, except that for ' mor- " ' tality profit ' we must substitute ' combined profit from " ' mortality, sickness and other benefits.' "

We have to thank Mr. Browne and Mr. Lidstone and the Council of the Institute of Actuaries for allowing us to reprint the articles.

Notes for a course of reading on "Valuation of Liabilities and Distribution of Surplus" (cp. *Students' Journal, Vol.* i., *No.* 2, *pp.* 51-52).

The following papers are specially suitable for a first reading on the subject:—

G. H. RYAN: J.I.A., XXXVIII., 69, &c.

G. J. LIDSTONE: J.I.A., XXXII., 73, &c. (including discussion).

T. G. C. BROWNE: J.I.A., XXXII., 194.

When reading these papers it should be remembered that conditions have changed somewhat since they were written, the rate of interest has risen and the British Offices' tables have come more into use. It is interesting and useful to read Sheppard Homans' early paper, J.I.A., xi., pp. 121, &c.

These papers are probably sufficient for an introduction to the subject, and the student should then study for himself the returns that have been made by Insurance Companies (*see* any recent volumes of the Board of Trade returns), noticing particularly the tables used, the rates of interest earned and assumed, the loading on premiums assumed in valuations for various classes of policies and the bonuses declared. A reference book such as Monilaws' "Surplus Funds" will assist.

A very useful exercise is to consider what would be the effect of a change in valuation basis on various Offices as regards surplus, loading, bonuses, &c. The tables given in J.I.A., xxxvii., 474-477, and J.I.A., xli., 50, 51, enable us

to make the necessary comparison of the reserves on various valuation bases for whole life and endowment assurances respectively.

It is also worth while to work out from these tables and the Board of Trade returns the effect on the reserves and surplus of different offices of (*a*) a large increase in new business and consequent increase in expenses; or, (*b*) a change in the rate of interest earned; or (*c*) a change in the rates of premium charged; or (*d*) a loss on investments.

Abstract of paper entitled "On the Equitable Distribution of Surplus," by SHEPPARD HOMANS, *appearing in Journal of the Institute of Actuaries, Vol.* xi., *pp.* 121, et seq.

[Date of paper: July, 1863.]

The experience of the Mutual Life Insurance Company of New York has been far more favourable, both in regard to mortality and interest, than was assumed when the premiums were determined on, or than it will be at all prudent to anticipate in future. Assuming a rate of mortality somewhat similar in the aggregate to that represented by the Carlisle table, and 4 per cent. interest, we have experienced during that time not more than 70 per cent. of that mortality and a net interest of $6\frac{1}{2}$ per cent. per annum, very nearly. The dividends are declared once in five years, or oftener, and the surplus is divided in accordance with a provision of the charter, which prescribes that the officers "shall credit each member with an *equitable share* of the profits."

A distribution of the large amount of surplus which must necessarily have arisen from conditions so favourable in proportion either to premiums paid or to sums assured, or, indeed, by any usual method, would have been far from equitable.

Let us assume in the first place the general correctness of the principle that each participant should be benefited in proportion to the excess of his payments over and above the actual cost of insurance. We shall find, however, that this principle is susceptible of some modification when a portion of the total surplus was due to sources other than the overpayments of present members. The first point, however,

after the amount of the general surplus shall have been correctly ascertained, will be to determine what the actual cost of insurance, and consequently the over-payment, has been in any and every case. Strictly speaking, this cannot be done until every policy shall have determined by death or otherwise; but, in that case, there could be no distribution of surplus during the "lifetime" of any policy. Practically, however, we can make such a reserve for all liabilities as to determine what sum may safely be distributed as surplus. The problem now before us, however, is not the amount but the equitable proportions of the surplus, which we assume really exists. As a general rule the accumulated payments in any policy are, or should be, absorbed—1st, by the actual cost of insurance (whatever that may have been); 2nd, by the reserve, and, 3rd, by the remainder, which is the over-payment or contribution to the surplus fund. The cost of insurance during any interval of time can only be determined by a careful analysis of the results of experience. The reserve being placed at the credit of the policyholder, will, even if too large, only increase by such excess his equitable share of surplus at the next distribution.

The following demonstration shows (in modern notation) how it is proposed to determine the contribution or over-payment of any policy:—

Let R_0 and R_1 be the valuation reserves for a policy (not necessarily upon the same basis) at the beginning and end of the year respectively.

Let i be the net rate of interest earned and P the net premium actually received by the office after allowing equitably for expenses and commission.

Let D_0 be the allowance applicable to the policy in respect of claims (assumed to be payable at the end of the year).

Then we have, assuming for simplicity that P is due at the beginning of the year,

$$(R_0+P)(1+i)-D_0-R_1=S_1,$$

the surplus in hand at the end of the year.

For a quinquennial period we have, similarly,

$$R_0(1+i)^5+P(s_{\overline{5}|}-1)-[D_0(1+i)^4+D_1(1+i)^3+\ldots+D_5]-R_5=S_5.$$

From the foregoing equations it appears that the contributions, or over-payments, of policies during a bonus period, may in general be found thus:—Credit each policy-holder, 1st, with the amount actually reserved at the last preceding distribution of surplus as the then present value of the policy; and, 2nd, with the effective premiums paid since that time, both sums being accumulated at the actual current rate of interest to the date of the present distribution; and charge him, 1st, with the actual cost of the risk to which the Company has been exposed during the interval, determined by means of a table representing the rates of mortality and interest actually experienced; and, 2nd, with the amount now reserved as the present value of the policy. The difference between the sum of his credits and the sum of his debits determines the over-payment or contribution from the policy proper.

If the amount of surplus at the previous distribution were determined on the assumption that the rates of mortality and interest would be less favourable than they have been since that time, a certain portion of the present surplus arising in consequence, clearly belongs exclusively to those persons who were insured previously to the date of that distribution, and who would be defrauded to a certain extent if more recent members were allowed to participate in that portion of the surplus, to which they could have no just claim. The contributions to surplus arising from a former dividend applied to the purchase of a reversionary addition of 1, or

from a paid up policy, or from a former dividend applied to the purchase of an annuity of 1 (acting as an annual reduction of premium), may be determined upon principles precisely similar to those already laid down.

The sum total of the contributions thus found showed what proportion of the general surplus resulted from the over-payments of existing members, and therefore what other portion resulted from extraneous sources, such as claim cases, lapses, &c. Nothing could be more just than that the portion first named should be placed at the credit of the original contributors in the proportions paid by each. This was done. The distribution of the remainder of the surplus, however, was not governed by the same principles of equity. It was a fund belonging to the members as a body, and might equitably have been divided amongst them all, *pro rata,* in equal proportions, or amongst them in proportion to premiums paid, or in proportion to sums insured, or, indeed, by any method which might have been prescribed by the charter or promised by the Company. Its distribution involved no principle of mathematical equity, and hence it should not necessarily be divided amongst policyholders in proportion to their contributions.

In the " Mutual Life " this portion of the general surplus was applied—1st, as an offset to the general expenses of conducting the business; 2nd, as a dividend of 25 per cent. upon participating premiums, on all policies where the contribution was less than such percentage; and, 3rd, the balance was divided amongst the remaining policyholders, in proportion to their contributions or over-payments.

It is believed that this is the first instance in which a Life Company has been enabled to determine what portion of the general surplus arose from the over-payments of participating members, and what other portion resulted from extraneous sources.

Paper on the Distribution of the Divisible Surplus of a Life Assurance Company, with special reference to the Method originated by Dr. Sprague and other Methods derived therefrom. By GEORGE J. LIDSTONE, *Fellow of the Institute of Actuaries.*

[Read before the Institute, 25th March, 1895.]

NOTE.—It should be remembered that the paper was written before the publication of the O^M and $O^{M(5)}$ Tables and the $O^{(M)}$ Table. The general argument based on the H^M and $H^{M(5)}$ Tables will usually apply to the O^M and $O^{M(5)}$ Tables, but the actual relations between the two sets of tables are quantitatively different.

INTRODUCTORY REMARKS.

1. Some time ago, the writer had occasion to examine the principal methods adopted by life assurance companies in the distribution of surplus, and particularly the plan which is known as "Sprague's Method," from the name of its inventor, Dr. Sprague, or the "Equity and Law Method," from the name of the company to which it was first applied. The plan is at present employed by comparatively few British offices, but it has recently been adopted by several companies in substitution for systems previously in vogue, and it appears to be rapidly growing in theoretical importance as a standard by which to test the effect of other systems of distribution. Nevertheless, it would appear that, apart from the brief description of the method originally given by Dr. Sprague (and quoted below) there is no systematic discussion of the principles of the method in the pages of the *Journal of the Institute of Actuaries*, and it is therefore thought that a few notes

respecting some of the points which arise in connection with the practical application of the method may be of service to those who are studying the subject, and who may share the experience that little information is to be gleaned from the *Journal*.

2. In the following remarks, it is not intended to give any strong expressions of opinion as to the exact system which should be followed in any special case, but rather to indicate and discuss some of the questions which are likely to arise for consideration. In actuarial work, it is almost impossible to lay down unbending rules which can be universally employed without modification, and this is especially the case with the subject of distribution of surplus, in reference to which our theoretical conclusions must often be modified by a variety of other considerations, arising out of the special circumstances of the office in question—such, for example, as the effect of methods which may have been followed in the past, and the representations which may have been made with regard to the future.

Dr. Sprague's Description of his Plan.

3. It will be desirable to quote at once, *in extenso*, Dr. Sprague's description of his plan. He says, in an editorial note (J.I.A., xiv., 396): "This appears a convenient oppor-"tunity for stating briefly the method of distribution of profits "which appears to ourselves to combine in the highest degree "the requisites of justice and facility of application, and "which we have found in practice to give very satisfactory "results.

"If the average rate of interest at which the total funds, "inclusive of bankers' and agents' balances, and other un-"productive assets, are improved, exceeds the rate at which

"the valuations are made, it is clear that a profit will be "realised in the nature of excess of interest on the amount "of the funds at the last valuation, to which profit the new "members have contributed nothing. Let the amount of "profit so earned by each of the old assured still remaining "on the books be ascertained and appropriated to his policy. "For example, if the valuations are made at 3 per cent. "interest, and the average rate realised has been £4 8*s*. per "cent., then the reserve made for each participating policy "at the last valuation (five years ago) is to be multiplied by ".0809 = $(1.044)^5 - (1.03)^5$; and the product will be the profit "to be in the first instance appropriated to the policy. The "sum of all these amounts being found and subtracted from "the surplus divisible among the assured, there will remain "a sum which may be fairly divided among all the assured "in proportion to the premiums (without interest) they have "respectively paid since the last valuation. As regards "persons of the same age at entry, it is clear that, apart "from the effect of selection, the profit on their current "premiums must be nearly the same whenever their policies "were effected; and if the premiums are loaded with a per-"centage on the net premium, or approximately so, the "distribution of the surplus in proportion to the premiums "paid will give very fair results. This method will have "the effect of giving larger cash bonuses to the policies the "longer they have been in force, but not unreasonably or "unfairly so."

4. It will be seen that the fundamental principle underlying the method is to make a broad division of the surplus into two parts, which are distributed according to different systems. The first part, consisting of interest earned in respect of the reserves under participating policies in excess of the rate assumed in the valuation, is divided in proportion

to the reserves upon which the excess interest has been earned. The second part embraces the whole of the remaining surplus (including the profit from loading, lapses, surrenders and other miscellaneous sources of surplus, increased or diminished by the profit or loss in respect of mortality), and is divided in proportion either to the tabular premiums paid, or (in some modifications of Sprague's plan) to the loadings paid. In its broad outlines, the method may thus be said to follow Mr. Sheppard Homans' "contribution method," which is based upon an elaborate analysis of the sources of profit (*see* J.I.A., xi., 121); but, while preserving to a great extent the advantages of that method, Dr. Sprague's plan avoids the complicated and somewhat cumbrous processes in consequence of which, and of certain theoretical objections, Mr. Homans' method has failed to receive much support from British actuaries.

Modification of Dr. Sprague's Plan.—The Loading System.

5. Several companies which have adopted the fundamental principles of Dr. Sprague's method of distribution have introduced changes of detail. Of these changes, perhaps the most important is that of substituting the loading for the office premiums as the measure for allotting the second portion of the surplus. It will be found that the practice of offices is equally divided on this point, for out of eight British offices which have adopted Dr. Sprague's principles, four adopt the loading system and four the premium system. If the loading be a constant percentage of the net premiums at all ages, the two methods will obviously give identical results, but this will not usually be the case in practice. Dr. Sprague himself, in his description of his method, says, "*If the premiums are loaded with a percentage*

" *on the net premiums, or approximately so,* the distribution " of the surplus in proportion to the premiums paid will give " very fair results," from which it seems fair to infer that if this condition were not fulfilled he would consider some modification necessary or desirable. This view is confirmed by some subsequent remarks by Dr. Sprague, appearing in his opinion on the case submitted to him by the Australian Mutual Provident Society. Mr. Black, the actuary of the society, had suggested the substitution of loading for premiums, and, commenting on this, Dr. Sprague said: " Considering the manner in which the premiums of the " society are adjusted . . . and the much heavier propor- " tionate loading at the younger ages, there seems little doubt " that the method proposed by Mr. Black is better suited for " adoption by the Society than that to which my name has " been attached."

Discussion of the Loading System.

6. Proceeding to consider the rationale of the loading method, we shall find it convenient to specify in detail the principal sources of surplus, which can be most conveniently exhibited as follows in the form of a

Profit and Loss Account.

1. Profit derived, in respect of participating policies, from interest on the reserves earned in excess of the rate assumed in the valuation.	1. Amount absorbed (if the valuations be made according to the H^M and $H^{M(5)}$ Tables) by passing from the H^M Reserves to the H^M and $H^{M(5)}$ Reserves in respect of business in its second quinquennium of policy existence.
2. Loading received in respect of participating policies.	2. Expenses of management and commission.
3. Profit from surrenders and lapses.	3. Miscellaneous losses.

4. The whole of the profit from the working of the non-participating business.	4. Loss from unfavourable mortality (if any).
5. Miscellaneous profits.	5. Surplus.
6. Profit from favourable mortality.	

7. The first two items of profit—namely, surplus interest and loading—can, without difficulty, be allotted to the several policies in the proportions in which they have been contributed.

8. The total of the items of profit classified under headings 3, 4 and 5, less miscellaneous losses under heading 3, represents what may be called "trading profit," which arises from the general working of the business, and can hardly be said to belong in any definite proportion to the continuing participating policyholders. On the other side of the account we have "expenses of management and commission," an item which it is also extremely difficult to apportion with anything approaching strict equity. Under these circumstances, it appears to be a suitable and convenient course to set off the miscellaneous profits under headings 3 to 5 against the cost of working the business and the miscellaneous losses—headings 2 and 3. The balance of profit or loss, as the case may be, will probably be relatively small (*see* paragraphs 15, 20 and 21), and may well be divided in proportion to the loading contributed, since that is primarily set aside as provision for expenses and contingencies, and therefore appears to be a fair measure for the allotment of the balance of profit or loss referred to.

9. The profit derived from favourable mortality will probably be due in a considerable measure to recency of selection (or as it is sometimes called "suspended mortality"), and if the valuation be made by the combined tables, we may reasonably apply such profit in the first instance to

provide for the amount absorbed, in respect of policies more than five but less than ten years in force, by passing from the H^M reserves to the H^M and $H^{M(5)}$ reserves.* The remainder of the profit from mortality (or the whole thereof if the valuation be made according to a single table) could, by a somewhat elaborate process, be divided among the participating policies, arranged in groups according to the attained age of the lives assured, in proportion to the difference between the expected death strain and the actual death strain experienced during the valuation period in respect of each group; but it would appear to be necessary, in order to avoid very undesirable irregularities in the resulting allotment of surplus, to introduce some graduation of the experienced rate of mortality before applying such a process. A method of this kind may be considered necessary when the mortality actually experienced differs very widely from that assumed in the valuation, as, for example, in the case of the Mutual of New York, which, at the time Mr. Sheppard Homans first introduced his "contribution" system of distribution, had experienced only 70 per cent. of the assumed rate of mortality. It may, however, be doubted whether so much refinement is necessary, or even desirable, in the more usual case when the table used in the valuation gives a fair representation of the rate of mortality likely to rule in the future,

* It may be of interest to remark, in passing, that the average amount of the additional reserve thus made, in the case of an office valuing at 3 per cent., will be about 1 per cent. on the sum assured. This may be shown by the use of Mr. King's hypothetical office. Thus, in the case of an office 10 years old, we find from Mr. King's tables:

Reserves H^M and $H^{M(5)}$ 3 per cent.	82,951	Number of policies in force in second quinquennium of existence as per table X (J.I.A., xx, 274)
Reserves H^M 3 per cent.	78,223	
Difference . . .	4,728 +	429,543

= 1·10 % nearly.

[The corresponding figures for O^M and $O^{M(5)}$ give about ·5 per cent.]

with perhaps a slight margin for contingencies. In such a case it would seem to be sufficient to divide the balance of mortality profit in proportion to the amounts of "death strain at risk," namely:

[Sums assured and bonuses] *minus* [Corresponding reserves].

The effect of this method will evidently be to give a smaller share of the surplus to policies of long duration than to policies recently effected, and the same result would be produced by the more elaborate plan referred to above unless the function

[Expected rate of mortality] *minus* [Actual rate of mortality]

increases more rapidly with the age than the quantity

[Death strain] ÷ [Amount at risk]

decreases. This result, while it may not be objectionable from a theoretical standpoint, will probably be considered undesirable in practice, especially in view of the fact that the expenses and commission press most heavily on the new business. An adjustment which, although rough, will be in the right direction may be obtained by dividing the balance of the profit from mortality, after making the initial deduction referred to in paragraph 8, according to some system which does not give a smaller share to policies of long duration than to policies of short duration. The desired result will be obtained if the balance of profit referred to be divided in proportion to the sums assured, the office premiums or the loading. It will not greatly matter which of these methods is selected, but it would be proper that the same plan which is adopted for the division of the mortality *profit* shall be used to apportion any *loss* which may arise from unfavourable

mortality in any valuation period. If the last method be adopted we shall allot the balance of mortality profit in the same proportion as the miscellaneous profits and losses, and we shall find that we have arrived at what has been called the "loading method," by reasoning which, although it may be said to be to some extent *a posteriori*, does not involve any great departure from principle.

Basis of Calculation of Loading.

10. In connection with the loading method, there arises for consideration a very important question which goes to the root of the whole matter, namely, on what basis should we calculate the net premium which is used in determining the loading ? The most obvious course, and that which appears to be commonly adopted in practice, is to employ the net premium which is used in the valuation. As far as the rate of interest is concerned, it certainly seems proper to adopt that used in the valuation, since the excess interest is returned in the same proportion in which it has been contributed, but it is not so clear that in other respects the valuation basis should be rigorously followed.

11. Take, for example, the common case of a company valuing its business by the combined H^M and $H^{M(5)}$ Tables. While this method practically gives very good results it is difficult to see that it has any scientific rationale, and it must be regarded as simply a convenient machinery for arriving at reserves which are very close to, but in most cases somewhat more stringent than, those which would be produced by the use of analysed or select tables. As Mr. King very forcibly and clearly puts it (J.I.A., xxxi., 251), "For valuation purposes an "aggregate table was of use only in so far as it gave a good "approximation to the reserves that would be made by the

" select tables. It was only because he found by practical " examination that the select tables prepared by Dr. Sprague " did very closely approximate to the result of the H^M and " $H^{M(5)}$ Tables that he used the latter combination. If they " had differed very much he should unhesitatingly have " thrown aside the combined tables and adopted the select." It is now generally recognised that the select tables give a more accurate measure of the risk premiums than is given by an aggregate table. It is therefore submitted that, in the division of surplus according to the method now under discussion, we shall obtain a more equitable apportionment as between the entrants at various ages by the use of the " select loading " than by the use of the " H^M loading," and if this be admitted, it seems evident that it is not right to disturb the whole incidence of the distribution merely because the valuation has not been made rigorously on the basis of the select tables, but by a method which gives a convenient approximation to the select reserves.

12. The same argument applies with slight modifications in the case of a company making its valuations on the basis of the H^M Table. The adoption of a basis giving weaker reserves than would be made by the use of the combined tables can have but very little influence on the real profit-earning power of the company, its principal effect being to throw a slightly larger amount into surplus during the early years of assurance and a correspondingly smaller amount during the middle and later years of assurance. In the case of a well-established company doing a normal amount of new business it may easily be seen that there is only a small difference between the amount of surplus periodically disclosed by a valuation according to the H^M Table and the corresponding amount which would be disclosed if the valuations were made throughout by the combined tables. For

example, if the office has reached a stationary condition, the new business being just sufficient to replace the "waste," the difference between the reserves will be constant and the annual interest on the difference will represent the amount by which the profit annually disclosed will differ from the corresponding amount in an office, similar in all other respects, but valuing by the H^M and $H^{M(5)}$ Tables. This will in any case bear but a small proportion to the total surplus, and it will be reduced in the event of the accession of a larger amount of new business than is required to maintain the office in a stationary position. It seems a fair conclusion that as the use of a weaker valuation basis makes comparatively small difference in the surplus it is not a sufficient reason for adopting the H^M loading in lieu of the select loading, a course which would involve a very considerable alteration in the relative amounts of surplus allotted to young and old entrants respectively.

13. It may be urged that if the select loading be adopted in the distribution we should, for the sake of consistency, use the select reserves as the basis of the allotment of the surplus interest, and that if this be not done our formulas for the cash surplus allotted to the various policies will be made up of heterogeneous elements. From a theoretical point of view it must be admitted that such an argument would be well founded, and if extended tables of policy-values, calculated strictly according to the select tables, were available it would certainly be desirable to employ them. As a matter of fact, however, such tables are not available, and we have to be content with approximate accuracy, and, under such circumstances, a much closer approximation will be obtained by the use of the hybrid formulas than by the use of the more symmetrical, but less correct, formulas involving the H^M loading.

Summary of the Results of the Loading System.

14. The operations of the loading method may be briefly summarised as follows:

(*a*) The profit derived from surplus interest is divided in proportion to the reserves in respect of which the interest has been earned.

(*b*) The whole of the loading paid in respect of each policy is next returned.

(*c*) The remainder of the profit derived from all sources is set against the expenses of management and commission and the losses of all kinds, the balance (profit or loss, as the case may be) being applied as a *pro rata* increase or decrease in the amount allotted under heading (*b*).

15. Now it will be seen that the premium method and the loading method practically agree in adopting the process of setting off the miscellaneous profits against the miscellaneous losses and the expenses of management, and it will be shown below that the difference (which is divided in the one case in proportion to the premiums paid, and in the other case in proportion to the loading) is, in the case of an average first-class office, comparatively small. There will thus be but little difference between the results of the premium method and the loading method in the case of the amount allotted under heading (*c*), the principal distinction between the two methods being that the latter returns all the loading in the proportions in which it has been contributed, while the former *throws the whole of the loading into one fund, which is then re-distributed in proportion to the premiums paid*. This course seems to be clearly inequitable (except in the unusual case of the premiums being loaded with a constant percentage, when the results of the two systems are coincident), and it would therefore appear that, to use the words of Dr. Sprague, the loading system is "better suited for adoption" than the premium system.

16. It may be desirable to consider somewhat more closely the reasoning of paragraph 8—to the effect that the miscellaneous profits on the one hand and the expenses of management and commission on the other, being very difficult to apportion between policyholders of different classes, a simple and not unfair way out of the difficulty is to set off one against the other. It will probably be admitted that the process in question is legitimate, so far as the general expenditure—excluding initial commission and expenditure, and all renewal commission—is concerned, but it may be said that the latter items are capable of being measured with approximate accuracy, and that the method of distribution should take account of this. In order to investigate this point, let us assume that the commission is at the common rate of 1 per cent. on the sum assured in respect of the first year's premium and $2\frac{1}{2}$ per cent. on renewal premiums, and that the initial expenditure, exclusive of commission, amounts to 1 per cent. on the sum assured. Spreading these amounts over the whole duration of the several policies, they will be equivalent to a uniform annual payment of

$$\frac{2+2{\cdot}5\pi a}{\mathbf{a}} = \frac{2-2{\cdot}5\pi}{\mathbf{a}} + 2{\cdot}5\pi = \mathrm{E}, \text{ say}$$

where π is the gross premium. The values of this expression will give for different ages at entry what has been called the "equalised pressure" of that portion of the expenditure with which we are now dealing, and they will form a fair basis for the apportionment of such expenditure.

17. In order to obtain numerical values of E, it will be necessary to fix upon a representative scale of office premiums, which will also be required in subsequent investigations. The following table exhibits for quinquennial ages rates which it is thought may be taken as fair average ones.

TABLE A.—*Showing the assumed Annual Rates of Premium for an Assurance of £100, with Profits.*

Age at Entry	Whole-Life Assurance π_x	Endowment Assurance, payable at 60 or at previous Death $\pi_{x:\overline{60-x}\vert}$	Age at Entry
20	1·950	2·450	20
25	2·175	2·850	25
30	2·450	3·400	30
35	2·800	4·150	35
40	3·225	5·300	40
45	3·800	7·200	45
50	4·550	11·050	50
55	5·525	..	55
60	6·900	..	60

18. Using the values of a according to the select tables at 3 per cent. we shall obtain, from the rates above given, the following table, showing the values of E for different ages at entry and also the values of the ratio

$$E \div \phi$$

where ϕ represents the loading on the same basis.

TABLE B.—*Showing the Annual Amount* (E) *equivalent to an initial expenditure of* 1 *per cent. with a commission of* 1 *per cent. on Sum Assured and* $2\frac{1}{2}$ *per cent. on Renewal Premiums; and the Ratio of that amount to the Total Loading* (ϕ).—(*Select* 3 *per cent.*).

Age at Entry	Whole-Life Assurances		Endowment Assurances at 60		Age at Entry
	E	$E \div \phi$	E	$E \div \phi$	
20	·136	·352	·156	·344	20
25	·144	·305	·171	·300	25
30	·155	·295	·193	·287	30
35	·169	·290	·223	·286	35
40	·187	·300	·268	·284	40
45	·210	·303	·342	·286	45
50	·240	·302	·486	·270	50
55	·278	·313	..	..	55
60	·332	·309	..	..	60

19. It will be seen that for whole-life assurances the ratio has a very nearly constant value of about 30 per cent., except in the case of entrants at the youngest ages, and that in the case of endowment assurances the ratios do not differ widely from that value. The results of dividing expenditure in proportion to the value of E will therefore differ only slightly from the results obtained by dividing it in proportion to ϕ.

From this it follows that the results which are obtained by the adoption of the principles of paragraph 8 are very close to those which would be obtained by the use of a more elaborate system, according to which an attempt would be made to apportion the expenditure among the various policies. and the miscellaneous profits would be divided in proportion to the loading.

The Average Results of First-Class Offices.

20. It has been stated above that in the case of an average first-class company the amount of profit or loss coming under heading (*c*) of paragraph 14 will be comparatively small, and that the result of the profit distribution is practically to return the whole of the loading calculated according to Sprague's "Select Tables" at the valuation rate of interest, in addition to the whole of the interest profit. This can be shown as follows: assuming the office to earn interest at the rate of $4\frac{1}{8}$ per cent. per annum, and to declare quinquennial profits representing on the average a compound reversionary bonus at the rate of £1 10*s*. per cent. per annum, with an interim bonus at the same rate, the gross premium should, on the hypothesis that the whole of the loading and surplus interest is returned in the shape of profits, be approximately equal to the net premium (calcu-

lated on the basis of select tables, with $4\frac{1}{2}$ per cent. interest) for the sum assured plus a compound reversionary bonus at £1 10*s*. per cent. per annum. The following table shows how nearly this is the case:

TABLE C.

Age at Entry	Whole-Life Assurance		Endowment Assurance at 60		Age at Entry
	Gross Premium	Net Premium on basis of Select Tables, with $4\frac{1}{2}$ per cent. Interest, for Sum Assured and Compound Reversionary Bonus of £1 10*s*. per cent. per annum	Gross Premium	Net Premium on basis of Select Tables, with $4\frac{1}{2}$ per cent. Interest, for Sum Assured and Compound Reversionary Bonus of £1 10*s*. per cent. per annum	
20	1·95	2·08	2·45	2·56	20
25	2·18	2·24	2·85	2·87	25
30	2·45	2·49	3·40	3·35	30
35	2·80	2·81	4·15	4·03	35
40	3·22	3·23	5·30	5·04	40
45	3·80	3·77	7·20	6·73	45
50	4·55	4·47	11·05	10·01	50
55	5·52	5·38	..	..	55
60	6·90	6·62	..	..	60

21. Dealing first with the whole-life assurances, the agreement is seen to be very close in the case of assurances effected at ages from 30 to 45, the ages at which policies are principally effected, but the younger entrants obtain rather more, and the older entrants rather less than the whole of the loading. In the case of endowment assurances it would seem that the effect of the compound reversionary bonus at 30*s*. per cent. per annum is to return rather less than the whole loading, so that, from the present point of view, endowment assurances appear to be treated less favourably than whole-life assurances.

Comparison of the Results of the Premium System and of the Loading System.

22. From the rates given in Table A the following table has been deduced, showing the loading and ratio of loading to gross premiums on four different assumptions as to the basis of calculation of the net or pure premiums:

Table D.—1. *Whole-Life Assurances.*

Age at Entry	Net Premium P_x	Loading ϕ_x	Ratio of Loading to Gross Preminm $\phi_x \div \pi_x$	Net Premium P_x	Loading ϕ_x	Ratio of Loading to Gross Premium $\phi_x \div \pi_x$	Age at Entry
	H^M 3 %			H^M 3½ %			
20	1·427	·523	·268	1·330	·620	·318	20
25	1.625	·550	·253	1·521	·654	·301	25
30	1·880	·570	·233	1·769	·681	·278	30
35	2·193	·607	·217	2.076	·724	·258	35
40	2·589	·636	·197	2·465	·760	·236	40
45	3·114	·686	·181	2·985	·815	·214	45
50	3·801	·749	·165	3·667	·883	·194	50
55	4·725	·800	·145	4·588	·937	·170	55
60	5·987	·913	·132	5·848	1·052	·152	60
	Sprague's Select Table 3 %			Sprague's Select Table 3½ %			
20	1·564	·386	·198	1·471	·479	·246	20
25	1·703	·472	·217	1·600	·575	·264	25
30	1·925	·525	·214	1·814	·636	·260	30
35	2·218	·582	·208	2·100	·700	·250	35
40	2·602	·623	·193	2·477	·748	·232	40
45	3·106	·694	·183	2.974	·826	·217	45
50	3·755	·795	·175	3·618	·932	·205	50
55	4·635	·890	·161	4.492	1·033	·187	55
60	5·826	1·074	·156	5·681	1·219	·177	60

TABLE D.—(*continued*).

2. *Endowment Assurances at* 60.

Age at Entry	Net Premium $P_{x:\overline{60-x}\rceil}$	Loading $\phi_{x:60-x\rceil}$	Ratio of Loading to Gross Premium $\phi_{x:\overline{60-x}\rceil} \div \pi_{x:\overline{60-x}\rceil}$	Net Premium $P_{x:\overline{60-x}\rceil}$	Loading $\phi_{x:\overline{60-x}\rceil}$	Ratio of Loading to Gross Premium $\phi_{x:\overline{60-x}\rceil} \div \pi_{x:\overline{60-x}\rceil}$	Age at Entry
	H^M 3 %			H^M 3½ %			
20	1·864	.586	·239	1·733	·717	·293	20
25	2·212	·638	·224	2·070	·780	·274	25
30	2·694	·706	·208	2·540	·860	·253	30
35	3·366	·784	·189	3·199	·951	·229	35
40	4·373	·927	·175	4·190	1·110	·209	40
45	6·060	1·140	·158	5·859	1·341	·186	45
50	9·389	1·661	·150	9·165	1·885	·171	50
	Sprague's Select Table 3%			Sprague's Select Table 3½ %			
20	1·997	·453	·185	1·871	·579	·237	20
25	2·281	·569	·200	2·141	·709	.249	25
30	2·727	·673	·198	2·571	·829	·244	30
35	3·371	·779	·188	3·203	·947	·228	35
40	4·357	·943	·178	4·173	1·127	·213	40
45	6.003	1·197	·166	5·801	1·399	·194	45
50	9.253	1·797	·163	9·028	2·022	·183	50

On consideration of the results of this table we shall arrive at the following conclusions:

(1) Assuming that the average whole-life premium in respect of policies in force at the date of distribution is about that for age 37,* and the

* This is not the same as the average premium in respect of new policies *effected*. It will be found that as the average duration of the policies increases the average premium decreases, because the entrants at the higher ages die off more rapidly than the younger entrants. The following figures, based on Mr. King's hypothetical office (J.I.A., xx, 233) will illustrate this point:

Age of Office Years	Number of Policies existing	Total Net Premium H^M 3 per cent.	Average Net Premium per cent.	Age to which Average Premium corresponds
0	127,471	3,092	2·425	38+
25	1,895,857	44,957	2·371	37−
50	2,515,624	57,498	2·286	36+

average endowment assurance premium about that for a 25 year term, the percentage of loading to gross premium is in the aggregate very nearly the same whether the H^M or select net premium be employed, but the average percentage is smaller for endowment assurances than for whole-life assurances. From this it follows that

(2) As between endowment assurances as a *class* and whole-life assurances as a *class*, approximately the same result would be obtained by the H^M loading plan and the select loading plan, and both would give a smaller proportion of surplus to the endowment assurances than would be given by the premium system.

(3) The ratio $\phi:\pi$ decreases as the age at entry increases, and therefore the loading system gives a larger proportion of the cash surplus divisible among whole-life assurances to policies effected on young lives, and also gives a larger proportion of the cash surplus divisible among endowment assurances to policies effected for a long term, than would respectively be allotted under the premium system.

(4) The features mentioned under heading (3) are not so strongly marked when the Select Tables are used to determine the net premium as when the H^M Table is used.

23. The above conclusions may be exhibited in a clearer light if we assume a representative rate of allotment, and thence determine the actual amount which would be allocated

(in addition to the interest profit) under the three different plans, to policies effected at various ages. The figures thus obtained will also serve as the basis of subsequent calculations, showing the amount of bonus addition which would be produced by the respective systems. Consistently with what has gone before it will be assumed :

(1) That the office premiums are those given in Table A.

(2) That the valuations are made on the basis of the combined H^M and $H^{M(5)}$ Tables, with interest at 3 per cent.

(3) That the average value of the ratio $\phi : \pi$ will be as follows : whole-life $H^M.207$; ditto, Select, .200 ; endowment assurances, $H^M.189$; ditto, Select, .188.

(4) That the participating whole-life assurances are to the participating endowment assurances in the ratio of 10 to 1.

(5) That the rate of interest earned is $4\frac{1}{8}$ per cent. per annum.

(6) That the distributions of profits are made annually, and that the total amount divided at the end of each year is equivalent to the whole of the interest profit on the reserve, plus the whole of the loading calculated according to the Select Tables at 3 per cent. interest.

24. From these data we shall obtain the figures given in Table E.

TABLE E.—*Showing the Cash Allotment to Policies effected at various ages, excluding the Interest Profit on Reserves.*

Age at Entry	Whole-Life Assurances			Endowment Assurances at 60			Age at Entry
	Premium System	H^M Loading System	Select Loading System	Premium System	H^M Loading System	Select Loading System	
20	·388	·507	·386	·488	·568	·453	20
25	·433	·533	·472	·567	·618	·569	25
30	·488	·552	·525	·677	·684	·673	30
35	·557	·588	·582	·826	·760	·779	35
40	·642	·616	·623	1·055	·898	·943	40
45	·756	·665	·694	1·433	1·105	1·197	45
50	·906	·726	·795	2·199	1·609	1·797	50
55	1·100	·775	·890	..	..	..	55
60	1·373	·885	1·074	..	..	..	60

25. It will be seen that the numerical results of Table E confirm the general conclusions of paragraph 22, and that, speaking generally, the results of the select loading system lie between those of the premium system and the H^M loading system. Comparing the results, age by age, it would appear that under the assumed conditions the premium system gives too small a proportion of surplus to policies subject to low rates of premium, and too large a proportion to policies subject to high rates of premium, while in the case of the H^M loading plan the conditions are reversed.

26. It has been laid down as one of the criteria of a good method of distribution of surplus that assuming the real profit-earning power of a company to remain unaltered, no considerable change in the amount of surplus allotted to policies effected at various ages and of different durations should be produced merely by a change in the relative proportions of the new business effected at different ages and under different tables. This must, perhaps, be regarded as an ideal result, which it is impossible in practice fully to secure,

but it nevertheless gives a good standard by which to judge the comparative merits of different methods. Applying this test to the premium system it is submitted that, under the supposed conditions, which have been chosen to represent as nearly as possible those which obtain in an average first-class company, the method does, to a certain extent, provide the bonuses allotted to policies effected on old lives and to endowment assurances having a short term to run out of the profit contributed by the younger entrants, and that, therefore, an increase in the proportion of new assurances belonging to the former classes will have the effect of diminishing the profits on policies of the latter class. The question is one of great and growing importance, having regard to the rapid increase, both absolute and proportional, which is taking place in the amount of assurances effected under the endowment assurance system, and more especially in view of the fact that such assurances are now frequently taken out for short terms by way of investment.

27. It seems hardly necessary again to point out—although to avoid misunderstanding it may be desirable to do so—that the above remarks will have to be very considerably modified if the premiums at the higher ages are greater than those given in Table A.

Distribution at Intervals Greater than One Year.

28. For the sake of convenience it has been assumed up to this point that the distribution of profits is made annually, and in theory this seems the most natural course to adopt. In practice, however, the valuation and division of surplus are most commonly made quinquennially, and it becomes desirable to consider what effect this may have upon the

method of distribution. Probably the chief reasons for the adoption of quinquennial instead of annual valuations are the following :

(1) By dealing with the profits earned over a period of five years we are likely to secure much more stability in the results of successive distributions than by dealing with the profits of individual years, which may show considerable fluctuations.

(2) The labour and expense attending a valuation and the consequent allotment of profits are so considerable that it is desirable, for the sake of practical convenience, to avoid having an annual distribution.

29. It seems proper, therefore, that the system of allotment adopted at the quinquennial distributions should be so adjusted as to produce results as nearly as possible equivalent to those which would be obtained if the profit were earned at a uniform rate during the quinquennium, and were actually distributed annually. This condition will be fulfilled if the present value at the beginning of any quinquennium of the profit to be allotted at the expiration thereof, plus the value of the interim bonus payable in the event of death during the quinquennium, be equal to the value of the profit which would be allotted in respect of the same period under the system of annual distribution. We shall thus have an equation of condition between the amounts of the quinquennial bonuses and the interim bonuses respectively.

30. There will be in practice three principal cases to consider :

(*a*) If no interim bonuses be allowed it will be proper to accumulate the annual cash allotment at

interest, with benefit of survivorship, in order to obtain the amount of cash surplus to be allotted at the end of the quinquennium.

(*b*) If the interim bonus allowed in case of death be the amount of *cash surplus* accrued since the last quinquennial distribution, the cash surplus to be allotted at the end of the quinquennium to those who survive will be found by accumulating the amount of cash surplus at interest only.

(*c*) If the interim bonus be the amount of bonus *addition* which would have accrued since the last quinquennial distribution, if the profits had been allotted and converted into bonus additions annually, the formulas for the periodical and interim bonuses become somewhat complicated, and they are therefore investigated in detail in Appendix A.

31. The principles of paragraph 29 will be most completely carried out by the adoption of method (*c*), which has accordingly been used in obtaining specimens of the amounts of the bonus additions which would respectively be allotted by the premium system, the H^M loading system and the select loading system, on the assumption that the annual cash surplus, in addition to the surplus interest, is that shown in Table E.

32. The specimen bonuses thus obtained are exhibited in the tables which are included in Appendix A. Perhaps the most noteworthy feature of the figures there given is the remarkable agreement between the results obtained, in respect of whole-life assurances, by the use of the H^M loading system and the uniform compound reversionary bonus system. The select loading plan, as would have been anticipated from the

reasoning of paragraphs 21 and 25, produces bonus additions which are smaller for the younger ages at entry and larger for the higher ages at entry than those which would be allotted according to the compound reversionary bonus plan. The tables thus confirm the results of previous investigations, which have shown that if the rate of interest realised be about 1 per cent. per annum in excess of that assumed in the valuation, the uniform bonus plan will give very satisfactory results, provided that the premiums be suitably adjusted; and it would appear that the requisite adjustment would be obtained by slightly increasing the premiums of Table A for the younger ages and decreasing them at the older ages.

33. Similar remarks will apply in the case of endowment assurances, but the tables do not confirm the view, which is held by some actuaries, that endowment assurances, as a class, are not entitled to as high a rate of bonus as that allotted to whole-life assurances.

Expressions for the Present Value of Future Allotments of Surplus.

34. It will be useful, for many purposes, to have expressions for the *present value* at any time of the surplus to be allotted in the future, on the assumption of any definite scale of cash allotment; such expressions will, for example, be of great assistance in comparing the average results obtained under different systems of distribution. In discussing this question, it will be assumed, for convenience, that the distributions are either made annually or are so adjusted as to produce results equivalent to those derived from an annual allotment.

35. There is no difficulty in obtaining the present value of that portion of the surplus which is divided in proportion

to the premiums or the loading. If a stationary rate of allotment be assumed, as for example $K\phi$ or $K\pi$ per annum, the value at any time will be $(K\phi)a$ or $(K\pi)a$, where a represents the annuity-value according to the table which is being used in the calculation, and if K be assumed to vary we shall have to take the value of a corresponding varying annuity which may be represented by the expression $\phi(Va)$ or $\pi(Va)$.

36. The formulas for the present value of the profit from surplus interest are much more complicated, and they might appear, at first sight, to be practically irreducible to a simple form. Nevertheless, it will be found on further investigation that the formulas for the present value (if the calculation be made on the basis of the "experience" rate of interest) reduce to the following simple and interesting form:

> [Valuation Reserve − Reserve according to experience rate of interest] + [present value of the difference between the premium used in the valuation and the corresponding premium according to the experience rate of interest.]

This result is perfectly general, and when once stated will no doubt appear to many actuaries to be sufficiently obvious from general considerations. Nevertheless, it appears to be desirable to demonstrate the formula algebraically, and the analysis, which will be found to be of an interesting character, is therefore given in detail in Appendix B, together with some further developments which may in some cases be practically useful.

Concluding Remarks.

37. The results given in the two preceding paragraphs might, it is thought, be of considerable use in the investiga-

tion of many important and interesting questions, such, for example, as the following: What is the effect of the system which divides the surplus in proportion to the premiums paid? How would the position of existing policyholders be affected by a change from such a system to one based upon Dr. Sprague's principles? What is the effect of a reduction in the rate of interest assumed in the valuation? It had originally been intended to touch upon these questions, but this paper has already extended to such length that any further discussion must be reserved for another occasion. The writer will therefore conclude by earnestly appealing to the senior members of the Institute, whose ripe experience peculiarly qualifies them to assist those who are studying one of the most important and difficult branches of actuarial science, to give the younger members the benefit of their views, not only upon the particular questions referred to in the preceding notes, but also upon the general subject of the distribution of the surplus of a life assurance company.

APPENDIX A.

Let us assume that the surplus interest earned is at the rate ρ per annum, and that the annual cash surplus allotted at the end of each year, in addition to the interest profit, is k per unit assured. Further, let

$V' = V + P$ = reserve in respect of sum assured alone immediately after payment of the premium.

B = bonus additions existing at the beginning of any quinquennium.

A = assurance value on valuation basis,

and let the subscripts 0, 1, 2 . . . respectively denote that the functions are calculated at the beginning of the quinquennium, one year later and so on. Then on the hypothesis that the cash surplus is allotted annually and is converted into reversionary bonus according to the valuation basis, the following table shows the total new bonuses (*i.e.*, excluding the bonus B) at the end of the 1st, 2nd . . . 5th year of the quinquennium.

Year	Bonus Addition arising from		
	(1) Surplus Interest on Policy Reserve	(2) Annual Cash Allotment of κ	(3) Surplus Interest on Reserve in respect of Bonus of B
	increased in each case by the Bonus arising from Surplus Interest on Profit from the same source in previous years of the quinquennium		
1	$\rho \frac{V'_0}{A_1}$	$\frac{\kappa}{A_1}$	$\rho B \frac{A_0}{A_1}$
2	$\rho \frac{V'_0}{A_1}\left(1+\rho\frac{A_1}{A_2}\right)+\rho\frac{V'_1}{A_2}$	$\frac{\kappa}{A_1}\left(1+\rho\frac{A_1}{A_2}\right)+\frac{\kappa}{A_2}$	$\rho B \frac{A_0}{A_1}\left(1+\rho\frac{A_1}{A_2}\right)+\rho B\frac{A_1}{A_2}$
⋮	⋮	⋮	⋮
5 (putting $1+\rho\frac{A_n}{A_{n+1}}=X_n$ &c.)	$\rho\left[\frac{V'_0}{A_1}X_1X_2X_3X_4+\frac{V'_1}{A_2}X_2X_3X_4+\frac{V'_2}{A_3}X_3X_4+\frac{V'_3}{A_4}X_4+\frac{V'_4}{A_5}\right]$	$\kappa\left[\frac{X_1X_2X_3X_4}{A_1}+\frac{X_2X_3X_4}{A_2}+\frac{X_3X_4}{A_3}+\frac{X_4}{A_4}+\frac{1}{A_5}\right]$	$\rho B\left[\frac{A_0}{A_1}X_1X_2X_3X_4+\frac{A_1}{A_2}X_2X_3X_4+\frac{A_2}{A_3}X_3X_4+\frac{A_3}{A_4}X_4+\frac{A_4}{A_5}\right]$

Since ρ is a small quantity and $\frac{A_x}{A_{x+1}}$ is very nearly equal to unity, we may, for practical purposes, substitute $1+\rho$ for X throughout, and if we further substitute five times the central value for the sum of five values, the results at the end of the fifth year may be approximately represented by the following formulas:

$$\left.\begin{array}{lll} \text{Column (1)} & \ldots & \frac{5\,(1+\rho)^2}{A_3}\,\rho V'_2 \\ \text{,, (2)} & \ldots & \frac{5\,(1+\rho)^2}{A_3}\,\kappa \\ \text{,, (3)} & \ldots & \frac{5\,(1+\rho)^2}{A_3}\,\rho B A_2 \end{array}\right\}$$

These formulas have been employed, in conjunction with the hypothetical rates of allotment referred to in paragraph 24 as the basis of calculation of the following tables showing the amounts of reversionary bonus added to policies of 1,000 according to the premium system, the H^M loading system, and the select loading system respectively. The tables are so arranged as to exhibit separately the bonus additions arising from each of the three sources of surplus (1, 2 and 3) referred to in the preceding tabular statement, and they also show the total bonuses and the results which would obtain if each quinquennial bonus were surrendered at the date of allotment, in which case there would be no profit under heading (3). For comparison, there has been added at the foot of each table a statement of the effect of a compound reversionary bonus of £1 10*s*. 0*d*. per cent. per annum.

In forming these tables for whole-life assurances, A has been taken from Sprague's Select Tables, V′ has been calculated on the basis of the H^M Table for the first quinquennium and the combined H^M and $H^{M(5)}$ thereafter. In the case of endowment assurances, both A and V′ have, for convenience, been taken from the H^M Table throughout.

Whole-Life Assurances—Age at Entry 20.

Bonus Additions to a Policy of 1,000 in respect of the under-mentioned Years of Assurance.

System of Allotment.	Profit arising under Heading	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40	41-45	46-50	51-55
Premium System	(1)	4·9	14·4	19·5	24·9	29·8	34·6	38·8	42·5	45·7	48·5	50·8
	(2)	54·2	50·9	46·9	43·0	39·4	36·0	33·0	30·3	28·1	26·2	24·6
	(1)+(2)	59·1	65·3	66·4	67·9	69·2	70·6	71·8	72·8	73·8	74·7	75·4
	(3)	..	3·4	7·2	11·4	15·9	20·7	25·9	31·4	37·4	43·7	50·6
	(1)+(2)+(3)	59·1	68·7	73·6	79·3	85 1	91·3	97·7	104.2	111·2	118·4	126·0
	Total Bonuses	59·1	127·8	201·4	280·7	365·8	457·1	554·8	659·0	770·2	888·6	1014·6
H^{M} Loading System	(1)	4·9	14·4	19·5	24·9	29·8	34·6	38·8	42·5	45·7	48·5	50·8
	(2)	70·8	66·5	61.3	56·2	51·5	47·0	43·1	39·6	36·7	34·2	32·1
	(1)+(2)	75·7	80·9	80·8	81·1	81·3	81·6	81·9	82·1	82·4	82·7	82·9
	(3)	..	4·3	9·1	14·2	19·6	25·3	31·3	37·8	44·6	51·9	59·6
	(1)+(2)+(3)	75·7	85·2	89·9	95·3	100·9	106.9	113·2	119·9	127·0	134·6	142·5
	Total Bonuses	75·7	160·9	250.8	346·1	447·0	553·9	667·1	787·0	914·0	1048.6	1191·1
Select Loading System	(1)	4·9	14·4	19·5	24·9	29·8	34·6	38·8	42·5	45·7	48·5	50·8
	(2)	53·9	50·6	46·7	42·8	39·2	35·8	32·8	30·2	27·9	26·0	24·5
	(1)+(2)	58·8	65·0	66·2	67·7	69·0	70·4	71·6	72·7	73·6	74·5	75·3
	(3)	..	3·3	7·2	11·3	15·8	20·6	25·7	31·3	37·2	43·6	50·4
	(1)+(2)+(3)	58·8	68·3	73·4	79·0	84·8	91·0	97·3	104·0	110·8	118·1	125·7
	Total Bonuses	58·8	127·1	200·5	279·5	364·3	455·3	552.6	656·6	767·4	885·5	1011·2
Compound Reversionary Bonus of 30*s.* per cent. per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	100·1	107·7	115·7	124·5	133·7	143·8	154·6
	Total Bonuses	75·0	155·6	242·3	335·5	435·6	543·3	659·0	783·5	917·2	1061·0	1215·6

Whole-Life Assurances—Age at Entry 30.

Bonus Additions to a Policy of 1,000 in respect of the under-mentioned Years of Assurance.

System of Allotment.	Profit arising under Heading	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40	41-45
Premium System	(1)	5·8	14·9	21·4	27·6	33·1	38·0	42·3	45·8	48·9
	(2)	59·0	54·1	49·6	45·3	41·5	38·2	35·3	32·9	30·9
	(1)+(2)	64·8	69·0	71·0	72·9	74·6	76·2	77·6	78·7	79·8
	(3)	..	3·7	7·8	12·2	17·0	22.2	27·9	33·9	40·3
	(1)+(2)+(3)	64·8	72·7	78·8	85·1	91·6	98·4	105·5	112·6	120·1
	Total Bonuses	64.8	137·5	216·3	301·4	393·0	491·4	596·9	709·5	829·6
H^M Loading System	(1)	5·8	14 9	21·4	27·6	33·1	38·0	42·3	45·8	48·9
	(2)	66·7	61·2	56·1	51·2	46·9	43·2	39·9	37·2	35·0
	(1)+(2)	72·5	76·1	77·5	78·8	80·0	81·2	82·2	83·0	83·9
	(3)	..	4·1	8·6	13·5	18·7	24·3	30·3	36·8	43·6
	(1)+(2)+(3)	72·5	80·2	86·1	92·3	98·7	105·5	112·5	119·8	127·5
	Total Bonuses	72·5	152·7	238·8	331·1	429·8	535·3	647·8	767·6	895·1
Select Loading System	(1)	5·8	14·9	21·4	27·6	33·1	38·0	42·3	45·8	48·9
	(2)	63·5	58·2	53·3	48·7	44·6	41·1	38·0	35·4	33·3
	(1)+(2)	69·3	73·1	74·7	76·3	77·7	79·1	80·3	81·2	82·2
	(3)	..	3·9	8·3	13·0	18·0	23·4	29·3	35·6	42·3
	(1)+(2)+(3)	69·3	77·0	83·0	89·3	95·7	102·5	109·6	116·8	124·5
	Total Bonuses	69·3	146·3	229·3	318·6	414·3	516·8	626·4	743·2	867·7
Compound Reversionary Bonus of 30s. per cent. per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	100·1	107·7	115 7	124·5	133·7
	Total Bonuses	75·0	155·6	242·3	335·5	435·6	543·3	659·0	783·5	917·2

Whole-Life Assurances—Age at Entry 40.

Bonus Additions to a Policy of 1,000 in respect of the under-mentioned Years of Assurance.

System of Allotment	Profit arising under Heading	1-5	6-10	11-15	16-20	21-25	26-30	31-35
Premium System	(1)	6·8	16·7	24·2	31·0	36·8	41·7	45·9
	(2)	65·2	59·5	54·6	50·2	46·4	43·3	40·7
	(1)+(2)	72·0	76·2	78·8	81·2	83·2	85·0	86·6
	(3)	..	4·1	8·6	13·6	19·0	24·8	31·1
	(1)+(2)+(3)	72·0	80·3	87·4	94·8	102·2	109·8	117·7
	Total Bonuses	72·0	152·3	239·7	334·5	436·7	546·5	664·2
H^M Loading System	(1)	6·8	16·7	24·2	31·0	36·8	41·7	45·9
	(2)	62·6	57·1	52·4	48·2	44·5	41·6	39·0
	(1)+(2)	69·4	73·8	76·6	79·2	81·3	83·3	84·9
	(3)	..	3·9	8·3	13·1	18·4	24·1	30·2
	(1)+(2)+(3)	69·4	77·7	84·9	92·3	99·7	107·4	115·1
	Total Bonuses	69·4	147·1	232·0	324·3	424·0	531·4	646·5
Select Loading System	(1)	6·8	16·7	24·2	31·0	36·8	41·7	45·9
	(2)	63·3	57·8	53·0	48·7	45·1	42·0	39·5
	(1)+(2)	70·1	74·5	77·2	79·7	81·9	83·7	85·4
	(3)	..	4·0	8·4	13·3	18·5	24·3	30·5
	(1)+(2)+(3)	70·1	78·5	85·6	93·0	100·4	108·0	115·9
	Total Bonuses	70·1	148·6	234·2	327·2	427·6	535·6	651·5
Compound Reversionary Bonus of 30s. per cent. per annum allotted Quinquennially	Quinquennial Bonus	75·0	80.6	86·7	93·2	100·1	107·7	115·7
	Total Bonuses	75·0	155·6	242·3	335·5	435·6	543·3	659·0

Whole-Life Assurances—Ages at Entry 50 and 60.

Bonus Additions to a Policy of 1,000 *in respect of the under-mentioned Years of Assurance.*

System of Allotment	Profit arising under Heading.	Age at Entry 50					Age at Entry 60		
		1-5	6-10	11-15	16-20	21-25	1-5	6-10	11-15
Premium System	(1)	8·2	19·0	27·6	34·7	40·8	10·2	21·9	31·5
	(2)	77·2	70·8	65·5	61·1	57·4	99·3	92·6	87·0
	(1)+(2)	85·4	89·8	93·1	95·8	98·2	109·5	114·5	118·5
	(3)	..	4·8	10·2	16·1	22·5	..	6·2	13·1
	(1)+(2)+(3)	85·4	94·6	103·3	111·9	120·7	109.5	120·7	131.6
	Total Bonuses	85·4	180·0	283·3	395·2	515·9	109·5	230·2	361·8
H^M Loading System	(1)	8·2	19·0	27·6	34·7	40·8	10·2	21·9	31·5
	(2)	61·8	56·8	52·5	49·0	46·0	64·0	59·7	56·1
	(1)+(2)	70·0	75·8	80·1	83·7	86·8	74·2	81·6	87·6
	(3)	..	4·0	8·5	13·5	19·1	..	4·2	9·1
	(1)+(2)+(3)	70·0	79·8	88·6	97·2	105·9	74·2	85·8	96·7
	Total Bonuses	70·0	149·8	238·4	335·6	441·5	74·2	160·0	256·7
Select Loading System	(1)	8·2	19·0	27·6	34·7	40·8	10·2	21·9	31·5
	(2)	67·7	62·2	57·5	53·6	50·4	77 7	72·5	68·0
	(1)+(2)	75·9	81·2	85·1	88·3	91·2	87·9	94·4	99·5
	(3)	..	4·3	9·1	14·5	20·4	..	5 0	10·6
	(1)+(2)+(3)	75·9	85·5	94·2	102·8	111·6	87 9	99·4	110·1
	Total Bonuses	75·9	161·4	255·6	358·4	470·0	87·9	187·3	297·4
Compound Reversionary Bonus of 30s. per cent. per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	100·1	75·0	80·6	86·7
	Total Bonuses	75·0	155·6	242·3	335·5	435·6	75·0	155·6	242·3

Endowment Assurances at 60—Age at Entry 20.

Bonus Additions to a Policy of 1,000 in respect of the under-mentioned Years of Assurance.

System of Allotment	Profit arising under Heading	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40
Premium System	(1)	6·2	14·7	22·6	29·9	36·6	42·9	48·6	53·8
	(2)	60·2	54·2	48·6	43·5	38·8	34·3	30·3	26·5
	(1)+(2)	66·4	68·9	71·2	73·4	75·4	77·2	78·9	80·3
	(3)	..	3·7	7·8	12·3	17·1	22·2	27·8	33·7
	(1)+(2)+(3) Total Bonuses	66·4	72·6	79·0	85·7	92·5	99·4	106·7	114·0
		66·4	139·0	218·0	303·7	396·2	495·6	602·3	716·3
H^M Loading System	(1)	6·2	14·7	22·6	29·9	36·6	42·9	48·6	53·8
	(2)	70·1	63·1	56·6	50·6	45·1	40·0	35·2	30·8
	(1)+(2)	76·3	77·8	79·2	80·5	81·7	82·9	83·8	84·6
	(3)	..	4·3	8·9	13·9	19·2	24·8	30·8	37·1
	(1)+(2)+(3) Total Bonuses	76·3	82·1	88·1	94·4	100·9	107·7	114·6	121·7
		76·3	158·4	246·5	340·9	441·8	549·5	664·1	785·8
Select Loading System	(1)	6·2	14·7	22·6	29·9	36·6	42·9	48·6	53·8
	(2)	55·9	50·3	45·2	40·4	36·0	31·9	28·1	24·6
	(1)+(2)	62·1	65·0	67·8	70·3	72·6	74·8	76·7	78·4
	(3)	..	3·5	7·4	11·6	16·2	21·1	26·6	32·2
	(1)+(2)+(3) Total Bonuses	62 1	68·5	75·2	81·9	88·8	95·9	103·3	110·6
		62·1	130·6	205·8	287·7	376·5	472·4	575·7	686·3
Compound Reversionary Bonus of 30s. per cent. per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	100·1	107·7	115·7	124·5
	Total Bonuses	75·0	155·6	242·3	335·5	435·6	543·3	659·0	783·5

Endowment Assurances at 60—Age at Entry 30.

Bonus Additions to a Policy of 1,000 in respect of the under-mentioned Years of Assurance.

System of Allotment	Profit arising under Heading	1-5	6-10	11-15	16-20	21-25	26-30
Premium System	(1)	7·6	18·2	27·9	37·0	45·2	52·9
	(2)	67·5	60·4	53·8	47·6	42·0	36·7
	(1)+(2)	75·1	78·6	81·7	84·6	87·2	89·6
	(3)	..	4·2	8·9	13·9	19·5	25·4
	(1)+(2)+(3) Total Bonuses	75·1	82·8	90·6	98·5	106·7	115·0
		75·1	157·9	248·5	347·0	453·7	568·7
H^M Loading System	(1)	7·6	18·2	27·9	37·0	45·2	52·9
	(2)	68·2	61·0	54·3	48·2	42·5	37·1
	(1)+(2)	75·8	79·2	82·2	85·2	87·7	90·0
	(3)	..	4·3	8·9	14·1	19·6	25·6
	(1)+(2)+(3) Total Bonuses	75·8	83·5	91·1	99·3	107·3	115·6
		75·8	159·3	250·4	349·7	457·0	572·6
Select Loading System	(1)	7·6	18·2	27·9	37·0	45·2	52·9
	(2)	67·1	60·0	53·4	47·4	41·8	36·5
	(1)+(2)	74·7	78·2	81·3	84·4	87·0	89·4
	(3)	..	4·2	8·8	13·9	19·4	25·3
	(1)+(2)+(3) Total Bonuses	74·7	82·4	90·1	98·3	106·4	114·7
		74·7	157·1	247·2	345·5	451·9	566·6
Compound Reversionary Bonus of 30s. per cent. per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	100·1	107·7
	Total Bonuses	75·0	155·6	242·3	335·5	435·6	543·3

Endowment Assurances at 60—Ages at Entry 40 and 50.

Bonus Additions to a Policy of 1,000 *in respect of the under-mentioned Years of Assurance.*

System of Allotment	Profit arising under Heading	Age at Entry 40				Age at Entry 50	
		1-5	6-10	11-15	16-20	1-5	6-10
Premium System	(1)	10·3	25·0	38·4	51·0	18·2	45·4
	(2)	83·8	74·3	65·5	57·2	136·5	119·2
	(1)+(2)	94·1	99·3	103·9	108·2	154·7	164·6
	(3)	..	5·3	11·1	17·4	..	8·7
	(1)+(2)+(3)	94·1	104·6	115·0	125·6	154·7	173·3
	Total Bonuses	94·1	198·7	313·7	439·3	154·7	328·0
H[M] Loading System	(1)	10·3	25·0	38·4	51·0	18·2	45·4
	(2)	71·3	63·2	55·7	48·7	99·9	87·2
	(1)+(2)	81·6	88·2	94·1	99·7	118·1	132·6
	(3)	..	4·6	9·8	15·5	..	6·6
	(1)+(2)+(3)	81·6	92·8	103 9	115·2	118·1	139·2
	Total Bonuses	81·6	174·4	278·3	393·5	118·1	257·3
Select Loading System	(1)	10·3	25·0	38·4	51·0	18·2	45·4
	(2)	74·9	66·4	58·5	51·1	111·5	97·4
	(1)+(2)	85·2	91·4	96·9	102·1	129·7	142·8
	(3)	..	4·8	10·2	16·1	..	7·3
	(1)+(2)+(3)	85·2	96·2	107·1	118·2	129·7	150·1
	Total Bonuses	85·2	181·4	288·5	406·7	129·7	279·8
Compound Reversionary Bonus of 30s. per cent. per annum allotted Quinquennially	Quinquennial Bonus	75·0	80·6	86·7	93·2	75·0	80·6
	Total Bonuses	75·0	155·6	242·3	335·5	75·0	155·6

APPENDIX B.

Demonstration of the Expression given in paragraph 36 for the Present Value of the Profit from Surplus Interest.

Taking first the simple case of a whole-life policy on a life aged x at entry, let

π = the valuation premium, which need not necessarily be the pure or net premium.

i = the valuation rate of interest.

${}_nV_x$ = the reserve at the expiration of n years.

Then ${}_nV_x + \pi$ will be the reserve at the commencement of the $(n+1)$th year immediately after payment of the premium, and will therefore be the amount on which the surplus interest is earned during the year.

Further, let the same symbols when accented denote quantities calculated according to the experience rate of interest, i' say.

Now ${}_nV_x + \pi = 1 - (1 + a_{x+n})(\pi + d) + \pi = v - a_{x+n}(\pi + d)$.

Thus, at the beginning of the $\overline{n+1}|$th year, the present value, calculated at the experience rate i', of the profit for the $\overline{n+1}|$th year will be

$$v'(i' - i)[v - a_{x+n}(\pi + d)]$$

and the present value of the profit for the $(n+2)$th year

$$v'^2(i' - i)[v - a_{x+n+1}(\pi + d)]\,{}_1p_x$$

and so on.

Summing, and introducing obvious abbreviations in the notation, we have

$$\Sigma = v'(i' - i)[v + v'v_1p + v'^2v_2p + \ldots - (\pi + d)(a + v'_1pa_1 + v'^2{}_2pa_2 + \ldots)]$$

$$= v'(i' - i)[(1 + a')v - (\pi + d) \times \text{B, say}] \quad \ldots\ldots \quad (1)$$

Now $B = v_1 p + v^2{}_2 p + v^3{}_3 p + \ldots$

$$+ v' v_2 p + v' v^2{}_3 p + \ldots$$

$$+ v'^2 v_3 p + \ldots$$

$$\ldots\ldots\ldots\ldots$$

$$= v_1 p + {}_2p(v'v + v^2) + {}_3p(v^3 + v^2 v' + v v'^2) + \ldots$$

which will be found to reduce to

$$\frac{v}{v - v'}(a - a') \quad \ldots\ldots\ldots \quad (2)$$

Substituting in (1), we obtain

$$\Sigma = v'(i' - i)\left[v(1 + a') - (\pi + d)(a - a')\frac{v}{v - v'}\right]$$

$$= vv'(i' - i)\left[1 + a' - (\pi + d)(a - a')\frac{1}{v - v'}\right] \quad \ldots \quad (3)$$

Now $vv'(i' - i) = vv'(\overline{1 + i'} - \overline{1 + i}) = v - v' = d' - d.$

Therefore expression (3) reduces to

$$(1 + a')(d' - d) - (\pi + d)(a - a')$$

$$= (1 + a')(\pi + d') - (1 + a)(\pi + d) \quad \ldots\ldots \quad (4)$$

$$= A - A' - [(1 + a) - (1 + a')]\pi$$

$$= {}_nV_x - {}_nV'_x + (1 + a'_{x+n})(\pi - \pi') \quad \ldots\ldots \quad (5)$$

where ${}_nV'_x = 1 - (1 + a'_{x+n})(\pi' + d')$.

Formula (5) expresses the present value, calculated at the *experience rate of interest*, of the profit from excess interest on the reserves, and from it the present value calculated at any other rate of interest may easily be found. Thus

$${}^{i}V - {}^{i'}V + {}^{i'}\mathbf{a}({}^{i}\pi - {}^{i'}\pi)$$

is the present value (at rate i') of interest profit at the rate of $i' - i$. Therefore

$${}^{i}V - {}^{i+k}V + {}^{i+k}\mathbf{a}\,({}^{i}\pi - {}^{i+k}\pi)$$

is the present value at rate $i+k$ of interest profit at rate k, and thus

$$\frac{i'-i}{k}\left[{}^{i}\mathrm{V}-{}^{i+k}\mathrm{V}+{}^{i+k}\mathbf{a}\left({}^{i}\pi-{}^{i+k}\pi\right)\right] \quad . \quad . \quad . \quad . \quad (6)$$

is the value at rate $i+k$ of interest profit at rate $i'-i$. The last expression may be written

$$(i'-i)\left[\frac{-\triangle_k \mathrm{V}-{}_{i+k}\mathbf{a}\,\triangle_k \pi}{k}\right]$$

(the symbol $\triangle_k$ denoting that the interval of differencing is k) where k may be positive or negative. If we put $k=0$ (*i.e.*, make the calculation at the valuation rate), the expression takes the form $\frac{0}{0}$, but its value is then obviously

$$(i'-i)\left[-\frac{d\mathrm{V}}{di}-{}^{i}\mathbf{a}\frac{d\pi}{di}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The form of expression (5) suggests that the result must be capable of considerable generalization, and we therefore proceed to investigate the most general case, introducing a varying rate of "experience" interest, and combining profit or loss from mortality with the profit or loss from interest.

Let i = the valuation rate of interest.

$i_1, i_2, i_3, \ldots\ldots$ = the rates of interest realized respectively in the 1st, 2nd, 3rd, . . . year.

$p_1, p_2, p_3, \ldots\ldots$ = the probability according to the valuation basis, that the status, if in existence at the beginning of the 1st, 2nd, 3rd, . . . year, will survive to the end of the year.*

* In the case of a compound status, $p+q$ will not necessarily be equal to unity.

$p'_1, p'_2, p'_3, \ldots$ = similar probabilities according to the rates of mortality actually experienced.

$q_1, q_2, q_3, \ldots$ = the probability, according to the valuation basis, that if the status be in existence at the beginning of the 1st, 2nd, 3rd, . . . year the sum assured will become payable at the end of the year.†

$q'_1, q'_2, q'_3, \ldots$ = similar probabilities according to the rates of mortality actually experienced.

$S_1, S_2, S_3, \ldots$ = the sum assured in the event of the contingency happening in the 1st, 2nd, 3rd, . . . year.

$\pi_1, \pi_2, \pi_3, \ldots$ = the premium payable at the commencement of the 1st, 2nd, 3rd, . . . year.

$B_n = vq_{n+1} S_{n+1} + v^2 p_{n+1} q_{n+2} S_{n+2} + \ldots$
= value of the benefit at the expiration of n years.

$P_n = \pi_{n+1} + vp_{n+1}\pi_{n+2} + v^2 p_{n+1} p_{n+2} \pi_{n+3} + \ldots$
= value of the future premiums at the expiration of n years.

$R_n = B_n - P_n$ = Reserve at the expiration of n years.

Then, at the commencement of the nth year, the present value of the interest profit receivable at the end of the year will be

$$v_n(i_n - i)(R_{n-1} + \pi_n) = v_n(i_n - i)(vp_n R_n + vq_n S_n)$$
$$= vv_n(i_n - i)(p_n R_n + q_n S_n) = (v - v_n)(p_n R_n + q_n S_n) \quad . \quad (8)$$

Similarly, at the commencement of the nth year, the present value of the mortality profit for that year will be

$$v_n(q_n - q'_n) S_n - v_n(p'_n - p_n) R_n \quad . \quad . \quad . \quad . \quad . \quad (9)$$

† In reckoning the years 1, 2, 3 . . . the starting-point is the date on which the present value is to be calculated, which need not be the date of the commencement of the assurance.

Adding expressions (8) and (9), we shall obtain

$$S_n(vq_n - v_n q'_n) + R_n(vp_n - v_n p'_n) \quad . \quad . \quad . \quad . \quad (10)$$

which must be multiplied by $v_1 v_2 \ldots v_{n-1} p'_1 p'_2 \ldots p'_{n-1}$ to obtain the present value at the beginning of the *first year*. Doing this and summing from $n = 1$ upwards, we have, after expanding R_n,

$$\begin{aligned} &S_1(vq_1 - v_1 q'_1) + (vp_1 - v_1 p'_1)\,[(vq_2 S_2 + v^2 p_2 q_3 S_3 + \ldots) \\ &\qquad - (\pi_2 + vp_2 \pi_3 + \ldots)] \\ &+ v'_1 p'_1 S_2 (vq_2 - v_2 q'_2) + v'_1 p'_1 (vp_2 - v'_2 p'_2)\,[vq_3 S_3 + v^2 p_3 q_4 S_4 + \ldots) \\ &\qquad - (\pi_3 + vp_3 \pi_4 + \ldots)] \\ &\qquad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ &\qquad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \end{aligned}$$

The coefficient of S_n in the total will be

$$\begin{aligned} &q_n\,[(vp_1 - v_1 p'_1)\, v^{n-1} p_2 \ldots p_{n-1} + v_1 p'_1\,(vp_2 - v_2 p'_2)\, v^{n-2} p_3 \ldots p_{n-1} \\ &\qquad + \ldots] + v_1 v_2 \ldots v_{n-1} p'_1 p'_2 \ldots p'_{n-1} (vq_n - v_n q'_n) \end{aligned}$$

which cancels out and reduces to

$$v^n p_1 p_2 \ldots p_{n-1} q_n - v_1 v_2 \ldots v_n p'_1 p'_2 \ldots p'_{n-1} q'_n.$$

Similarly, the coefficient of π_n will be

$$\begin{aligned} &- [(vp_1 - v_1 p'_1) v^{n-2} p_2 p_3 \ldots p_{n-1} + v_1 p'_1 (vp_2 - v_2 p'_2) v^{n-3} p_3 \ldots p_{n-1} \\ &\qquad + \ldots + v_1 v_2 \ldots v_{n-2} p'_1 p'_2 \ldots p'_{n-2} (vp_{n-1} - v_{n-1} p_{n-1})] \\ &= -(v^{n-1} p_1 p_2 \ldots p_{n-1} - v_1 v_2 \ldots v_{n-1} p'_1 p'_2 \ldots p'_{n-1}). \end{aligned}$$

Thus the whole expression reduces to

$$\begin{aligned} &[S_1(vq_1 - v_1 q'_1) + S_2(v^2 p_1 q_2 - v_1 v_2 p'_1 q'_2) + \ldots] \\ &\qquad - [\pi_1 (1 - 1) + \pi_2 (vp_1 - v_1 p'_1) \ldots] \\ &= R_0 - R'_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11) \end{aligned}$$

where R'_0 indicates that the reserve is calculated at the experience rates of mortality and interest, the premiums valued remaining unchanged. If, then, R''_0 represent the reserve on the experience

basis, *assuming the premiums payable to be different from those employed in obtaining* R_0, we shall have the most general expression

Present value of profit from mortality and interest

$$= R_0 - R_0'' + [\text{present value, at experience rates, of the difference between the premiums used in the valuation and the corresponding premiums used in determining } R''_0] \quad . \quad . \quad (12)$$

From this general expression we may deduce the results for any particular class of benefit. Thus, putting $S_1 = S_2 = S_3 \ldots$, $\pi_1 = \pi_2 = \pi_3 \ldots p_1 + q_1 = p'_1 + q'_1 = p_n + q_n \ldots = 1$, we have the case of an ordinary whole-life assurance; again, putting $\pi_1 = \pi_2 \ldots = 0$, we have the case of a paid-up whole-life assurance, and the formula reduces to $A_o - A'_o$. For annuities, we must put $q_o = p_o$, $q'_o = p'_o \ldots$, and the expression becomes $a_o - a'_o$, and so on.

In the discussion GEORGE KING gave the following demonstration:

Let Company A value at rate of interest i, and let the net premium on that basis be P_1, and let the policy-value be represented by V. Let Company B value by a higher rate j, with corresponding net premium P_2, and policy-value U. We then have the self-evident equations

$$\text{A.} \quad (V + P_1)(1+i) = q + pV_{+1}$$

$$\text{B.} \quad (U + P_2)(1+j) = q + pU_{+1}$$

which simply tell us that the reserve at the beginning of the year, with the net premium then paid, accumulated at the valuation rate of interest, is equal to the claims of the year and the reserve at the end of the year. Now, assume for the moment that both companies realize rate j. Then Company A by making the higher reserves loses in the year, as compared with B, loading to the

amount of $(P_1-P_2)(1+j)$, and gains in interest $(V+P_1)(j-i)$. In the early days of a policy the loss will be greater than the gain, and A will have the smaller divisible surplus, while later on the position will be reversed. The two companies will be on a par in this respect when $(P_1-P_2)(1+j)=(V+P_1)(j-i)$, that is, when $V(j-i)=P_1(1+i)-P_2(1+j)$, that is, when

$$V=\frac{P_1(1+i)-P_2(1+j)}{j-i}.$$

By calculating out the expression for any age at entry, and referring to a table of policy-values, the time of equal surplus can at once be found. Thus, taking age 35 and rates of interest 3 per cent. and 4 per cent., and using the H^M Table, we find that, at the moment of equality of surplus, $V=\cdot211$, corresponding to a duration of between 12 and 13 years of the policy. For 12 years, Company A, making the more stringent valuation, will have the smaller surplus, but after 13 years it will have the greater. For younger ages at entry the time will be longer, and for older ages shorter. Taking now the more general case of both companies valuing below the rate of interest realized, which may be written J, so that $J>j>i$, with Company A the loss of loading will be $(P_1-P_2)(1+J)$, and the gain from interest will be $(V+P_1)(J-i)$, so that the net gain or loss, as the case may be, will be

$$\begin{aligned}&(V+P_1)(J-i)-(P_1-P_2)(1+J)\\&=V(J-i)-P_1(1+i)-P_2(1+J).\end{aligned}$$

With Company B there will be no loss of loading, and the gain from interest will be $(U+P_2)(J-j)$. The companies will be on an equality when these expressions are equal, that is, when

$$V(J-i)-P_1(1+i)-P_2(1+J)=U(J-j)+P_2(J-j),$$

that is, when

$$V(J-i)-U(J-j)=P_1(1+i)-P_2(1+j).$$

In this more complicated case the time cannot be found by simple reference to a table of policy-values, but by making trials it can

be ascertained by interpolation. Thus, taking age 35 as before, and rates of interest 4 per cent., 3 per cent. and $2\frac{1}{2}$ per cent., and assuming first 10 years, we have

$$V(J-i)-U(J-j)=\cdot00093,$$
$$P_1(1+i)-P_2(1+j)=\cdot00121,$$

so that at the end of 10 years Company A still yields less surplus. At the end of 15 years $V(J-i)-U(J-j)=\cdot00142$, so that now A yields the larger surplus, and by interpolation we find that the point of equality is reached between 12 and 13 years.

G. F. Hardy suggested that in arriving at interest profits they should make a deduction for expenses. If they had a large fund to invest, it would be impossible to conduct the business without a considerable expenditure, and that portion of the expenses of management which was due to the investing of the funds must be in many offices considerable, and should be taken into account in deciding what were the profits from interest. An office, for example, which was making $4\frac{1}{2}$ per cent. on its funds might very reasonably set aside at least $\frac{1}{4}$ per cent. representing the cost of gaining that surplus rate of interest. With regard to the result in § 36 he remarked that taking the case of one company earning a rate of interest i' and valuing at that rate, and another company earning the same rate of interest and valuing at the rate i, the difference between the sums they held in hand at any time as to any policy must be the expression Mr. Lidstone had put down, $R_0-R''_0$. The difference between the sums the two offices would hold, must evidently be equivalent to the bonus which would be distributed in future to the policyholder of the office valuing at the lower rate of interest. Putting it in that way might bring out a little more strongly the verbal meaning of the result the author arrived at algebraically.

The Guardian Method of Division of Profits. [*Extracted from a letter addressed by* T. G. C. BROWNE *to The Insurance Record of 19th July*, 1895, *and reprinted in the Journal of the Institute of Actuaries, Vol.* xxxii., *pp.* 194, et seq.]

I now beg to send you the analysis of the net profit actually divided among the policyholders during the 20 years 1875-1894 inclusive, and I venture to think that it is of some general interest from the principles it illustrates. The profit is divided into three groups, as described in the following statement, and Group I increased or decreased by its rateable proportion of Group III is divided according to the loading received during the five years; while Group II similarly increased or decreased, is divided according to the reserve value at the *end* of that period. This method of division was applied consistently to all classes of policies entitled to participate. In the following statement the various items of profit included in Group III are given separately, as it is upon the mode of dealing with them that discussion is most likely to arise. The sign + is prefixed to a profit and − to a loss:—

[*See next page.*]

In looking at the figures for 1875-1879, at the end of which period my method was first adopted, the prominent feature is the very large loss from mortality, and it was apparent that neither Dr. Sprague's method nor that of the Australian Mutual Provident could be used satisfactorily under the circumstances. I had therefore to consider what modifications were necessary to make the latter (which I preferred)

	1875-9	1880-4	1885-9	1890-4
GROUP I:				
Loading, less Expenses and Commission, Surrenders, and Lapses . . .	+ 36,454	+ 49,443	+ 62,719	+ 63,587
GROUP II:				
Interest earned above 3 per cent., including Interest earned on Reversions, at an assumed rate . .	+ 81,941	+ 88,986	+ 85,907	+ 79,144
GROUP III:				
(1) Mortality . . .	− 29,282	+ 6,411	+ 47,306	+ 48,620
(2) Investments realized, including the profit from Reversions, less the assumed rate of Interest included in Group II .	+ 6,487	+ 7,414	− 1,474	+ 10,811
(3) Bonuses on Re-assurances	+ 4,596	+ 7,278	+ 13,912	+ 12,784
(4) Interest on Investment Reserve Fund, undivided Balance from preceding period, and on accrued profit . .	+ 11,804	+ 14,283	+ 15,762	+ 22,188
(5) Issue Policies . . .	..	+ 1,413	+ 3,802	+ 2,560
(6) Annuities* . . .	..	..	− 1,511	+ 3,436
Total of Group III .	− 6,395	+ 36,799	+ 77,797	+100,399
Total amount divided among Policyholders in force at end of period	+112,000	+175,228	+226,423	+243,130
Loading Multiplier . . .	·461	·525	·740	·772
Reserve " . . .	·0733	·0764	·0773	·0736

NOTE.—The business of the London and Provincial Law Office was taken over in 1882—hence the large increase in the figures dealt with.

*These were kept until recent years in a separate fund.

applicable to the case before me. The creation of a third group to be rateably divided between the other two, appeared to me absolutely necessary in order to make a contribution method work satisfactorily in times of adversity as well as of prosperity, and I had to decide how that group should be

composed. The peculiar circumstances of the quinquennium may have influenced me in the choice of the items to be included in it, but I have seen no reason to alter it during the succeeding three periods, although a modification, which I shall refer to presently, may be desirable hereafter.

When the object of a third group is kept in view, the principle of its composition is obviously that it should include all important items which are likely to fluctuate greatly from period to period, thus disturbing the practical working of the system. The various items of Group III, I think, fairly comply with these conditions, except one—"Bonuses on Re-assurances," which I included simply for the sake of convenience. It represents the present cash value of the total bonuses on re-assurances declared during the quinquennium, and strictly it is not a profit at all, except so far as it exceeds the bonuses declared on the re-assured portion of the principal policies. To bring these bonuses into account year by year as they are declared and to set them against the principal policies, would involve a considerable amount of trouble, and they have therefore been dealt with in a lump sum at the end of the quinquennium as an item in Group III. Dr. Sprague states that he is opposed to my views as to the treatment of profit or loss from investments, and he argues that this item should be treated as an addition to or deduction from interest. Theoretically I agree with him, and if such an item occurred with regularity in one direction only, there would be no objection to treating it as he suggests, but unfortunately an examination of the accounts of many companies shows that this item is perhaps the most variable that the actuary has to deal with; I contend therefore that if there is a necessity for a third group at all it should be included in it. I take a similar view of the profit from reversions above an assumed rate of interest.

In the foregoing statement, I give the Loading and Reserve Multipliers for the four periods; thus, for 1880-84 these are .525 and .0764 respectively, that is to say, a little more than one-half of the loading and upwards of $1\frac{1}{2}$ per cent. per annum on the reserve were apportioned in cash to the policyholders, who received the full cash apportionment without any deduction, or a corresponding reversion on an H^M or $H^{M(5)}$ 3 per cent. basis.

The following specimens of the reversionary bonuses declared at the four divisions during which the profits earned have varied greatly, is the best test of my method as a practical one :

Age at Entry	Year	Number of Years in Force								
		5	10	15	20	25	30	35	40	45
		£ s.	£ s.	£ s.	£ s.	£ s.	£ s.	£ s.	£ s.	£ s.
30	1879	4 8	5 6	6 2	6 18	7 14	8 0	8 13	9 6	10 0
	1884	5 6	6 4	7 4	8 2	8 18	9 14	10 0	10 14	11 6
	1889	6 16	7 14	8 12	9 10	10 6	11 0	11 14	11 18	12 10
	1894	*6 2	7 18	8 14	9 12	10 10	11 4	11 18	12 12	12 14
40	1879	4 1	5 2	6 2	7 1	7 19	8 8	9 3	10 0	10 19
	1884	4 15	5 17	7 0	7 19	8 18	9 16	10 2	10 19	11 15
	1889	6 2	7 4	8 6	9 8	10 8	11 4	12 2	12 8	13 4
	1894	6 2	7 6	8 6	9 6	10 8	11 4	12 2	12 18	13 4

* The decline in this bonus as compared with 1889 is accounted for by a reduction in the rate of premium at age 30.

At the end of the last quinquennium, in consequence of the continuous fall in the rate of interest, it was decided to carry forward a much larger balance of undivided profit than on previous occasions instead of lowering the valuation rate of interest. Under these circumstances, I think that in future the interest on this balance might be included in Group II, together with interest on any investment reserve fund which may exist, provided that such fund has been created purely as a general precautionary measure and not to meet specific losses actually in view.

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